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Some Unitary Operators Derived Using the Quantum State and Its Application

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Abstract Some unitary operators are derived using quantum state by depending on the technique of integration within an ordered product of operators, for example parity operator, displacement operator, squeezed operator, etc. The characteristics of these operators are analyzed. Their unitary transformations play an essential role in some transformations. As applications, the dynamic problems of the double momentum coupling harmonic oscillators are solved exactly.

Keywords Unitary operators · IWOP technique · Quantum state

1 Introduction

The representation theory and its transformation have proved very useful in studying quantum mechanics, quantum optics and quantum statistics. The unitary transformations play an essential role in space inversion, coordinate translation, squeezing transformation, canonical transformation and symplectic transformation. The authors of [1–13] had studied the representation theory and its transformation widely. Those works have greatly developed representation and transformation theory. Fan has created the technique of integration within an ordered product (IWOP) of operators [14, 15], which has developed Dirac's symbol method and representation theory greatly. References [14, 15] had given the eigenvector $|x\rangle$ of the coordinate operator $X=(\hat{a}+\hat{a}^{\dagger})/\sqrt{2}$

$$|x\rangle=\pi^{-1/4}\exp(-x^2/2+\sqrt{2}x\hat{a}^{\dagger}-\hat{a}^{\dagger 2}/2)|0\rangle, \tag{1}$$

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and the eigenvector $|p\rangle$ of the momentum operator $P = (\hat{a} - \hat{a}^\dagger)/\sqrt{2}\mathrm{i}$

$$|p\rangle = \pi^{-1/4} \exp(-p^2/2 + \mathrm{i}\sqrt{2} p \hat{a}^\dagger + \hat{a}^{\dagger 2}/2)|0\rangle. \tag{2}$$

In this paper, some unitary operators are derived using the eigenvector $|x\rangle$ and $|p\rangle$ by depending on IWOP technique. The characteristics of these operators are analyzed. As applications, the dynamic problems of the double momentum coupling harmonic oscillators are solved exactly.

2 Some Unitary Operators

2.1 Parity Operator

Based on the space inversion $\vec{r} \to -\vec{r}$, we construct the following asymmetric integration form projection operator

$$\hat{P} = \int_{-\infty}^{\infty} dx |-x\rangle\langle x|. \tag{3}$$

Using the normal product form of vacuum projection operator $|0\rangle\langle 0| =: \exp(-\hat{a}^\dagger \hat{a}):$ and integral formula $\int_{-\infty}^{\infty} dx \exp(-\alpha x^2 + \beta x) = \sqrt{\frac{\pi}{\alpha}} \exp(\frac{\beta^2}{4\alpha})$, we obtain

$$\begin{aligned}
\hat{P}\hat{P}^\dagger &= \iint_{-\infty}^{\infty} dx dx' |-x\rangle\langle x||x'\rangle\langle -x'| \\
&= \iint_{-\infty}^{\infty} dx dx' |-x\rangle \delta(x - x')\langle -x'| = \int_{-\infty}^{\infty} dx |-x\rangle\langle -x| \\
&= \pi^{-1/2} \int_{-\infty}^{\infty} dx \exp(-x^2/2 - \sqrt{2} x \hat{a}^\dagger - \hat{a}^{\dagger 2}/2)|0\rangle\langle 0| \exp(-x^2/2 - \sqrt{2} x \hat{a} - \hat{a}^2/2) \\
&= \pi^{-1/2} \int_{-\infty}^{\infty} dx \exp(-x^2/2 - \sqrt{2} x \hat{a}^\dagger - \hat{a}^{\dagger 2}/2) : \exp(-\hat{a}^\dagger \hat{a}) : \\
&\quad \times \exp(-x^2/2 - \sqrt{2} x \hat{a} - \hat{a}^2/2) \\
&= \pi^{-1/2} \int_{-\infty}^{\infty} dx : \exp[-x^2 - \sqrt{2} x (\hat{a}^\dagger + \hat{a}) - (\hat{a}^\dagger + \hat{a})^2/2] := 1 = \hat{P}^\dagger \hat{P},
\end{aligned}$$

where :: denotes normal ordering. We can see that $\hat{P}$ is unitary operator.

From (3), the transformation property of $\hat{P}$ for $\hat{x}$ and $\hat{p}$ can be obtained.

$$\begin{aligned}
\hat{P}\hat{x}\hat{P}^\dagger &= \iint_{-\infty}^{\infty} dx dx' |-x\rangle\langle x|\hat{x}|x'\rangle\langle -x'| = \iint_{-\infty}^{\infty} dx dx' |-x\rangle x' \delta(x - x')\langle -x'| \\
&= \int_{-\infty}^{\infty} x dx |-x\rangle\langle -x| \\
&= \pi^{-1/2} \int_{-\infty}^{\infty} dx : x \exp[-x^2 - \sqrt{2} x (\hat{a}^\dagger + \hat{a}) - (\hat{a}^\dagger + \hat{a})^2/2] := -\hat{x},
\end{aligned}$$

$$\hat{P}\hat{p}\hat{P}^{\dagger} = \iint_{-\infty}^{\infty} dxdx'|-x\rangle\langle x|\hat{p}|x'\rangle\langle -x'| = \iint_{-\infty}^{\infty} dxdx'\left[\mathrm{i}\frac{\partial}{\partial x'}|-x\rangle\delta(x-x')\right]\langle -x'|$$

$$= \mathrm{i}\pi^{-1/2}\int_{-\infty}^{\infty} dx : (-x-\sqrt{2}a^{\dagger})\exp[-x^2-\sqrt{2}x(\hat{a}^{\dagger}+\hat{a})-(\hat{a}^{\dagger}+\hat{a})^2/2] := -\hat{p},$$

where we have used the integral formula $\int_{-\infty}^{\infty} xe^{-\alpha x^2+\beta x}dx = \frac{\beta}{2\alpha}\sqrt{\frac{\pi}{\alpha}}\mathrm{e}^{\frac{\beta^2}{4\alpha}}$. We can see that $\hat{P}$ is parity operator.

The explicit forms of $\hat{P}$ can be derived by depending on IWOP technique

$$\begin{aligned}
\hat{P} &= \int_{-\infty}^{\infty} dx|-x\rangle\langle x| \\
&= \pi^{-1/2}\int_{-\infty}^{\infty} dx\cdot\exp\left(-x^2-\sqrt{2}x\hat{a}^{\dagger}-\frac{\hat{a}^{\dagger 2}}{2}\right) : \exp(-\hat{a}^{\dagger}\hat{a}) : \exp\left(\sqrt{2}x\hat{a}-\frac{\hat{a}^2}{2}\right) \\
&= \pi^{-1/2}\int_{-\infty}^{\infty} dx : \exp\left[-x^2+\sqrt{2}x(\hat{a}-\hat{a}^{\dagger})-\frac{(\hat{a}+\hat{a}^{\dagger})^2}{2}\right] :=: \exp(-2\hat{a}^{\dagger}\hat{a}) : \\
&=: \exp[(e^{\mathrm{i}\pi}-1)\hat{a}^{\dagger}\hat{a}] := \exp(\mathrm{i}\pi\hat{a}^{\dagger}\hat{a}) = (-1)^{\hat{a}^{\dagger}\hat{a}} = (-1)^{\hat{N}},
\end{aligned}$$

where we have used the formula $:\exp[(e^{\lambda}-1)\hat{a}^{\dagger}\hat{a}] := \exp(\lambda\hat{a}^{\dagger}\hat{a})$. So

$$\hat{P}|n\rangle = (-1)^n|n\rangle.$$

The Fock state is the eigen state of $\hat{P}$.

In the case of multimode, the unitary operator can be constructed by following asymmetric integration

$$\hat{P} = \int_{-\infty}^{\infty} dx_1dx_2\cdots dx_n|-x_1,-x_2,\ldots,-x_n\rangle\langle x_1,x_2,\ldots,x_n|. \tag{4}$$

It is easily proved that

$$\begin{aligned}
\hat{P}\hat{P}^{\dagger} &= \iint_{-\infty}^{\infty} dx_1\cdots dx_ndx_1'\cdots dx_n'|-x_1,\ldots,-x_n\rangle\langle x_1,\ldots,x_n||x_1',\ldots,x_n'\rangle \\
&\quad\times\langle -x_1',\ldots,-x_n'| \\
&= \iint_{-\infty}^{\infty} dx_1\cdots dx_n|-x_1,\ldots,-x_n\rangle\langle -x_1,\ldots,-x_n| = 1 = \hat{P}^{\dagger}\hat{P}.
\end{aligned}$$

Obviously, $\hat{P}\hat{x}_i\hat{P}^{\dagger} = -\hat{x}_i$, $\hat{P}\hat{p}_i\hat{P}^{\dagger} = -\hat{p}_i$.

2.2 Displacement Operator

Based on the displacement transformation of classical physics $x \to x - x_0$, we construct the following asymmetric integration form projection operator

$$\hat{D} = \int_{-\infty}^{\infty} dx|x-x_0\rangle\langle x|. \tag{5}$$

It is easily proved that

$$\hat{D}\hat{D}^{\dagger}=\iint_{-\infty}^{\infty}dxdx'|x-x_0\rangle\langle x||x'\rangle\langle x'-x_0|=\int_{-\infty}^{\infty}dx|x-x_0\rangle\langle x-x_0|=1=\hat{D}^{\dagger}\hat{D}.$$

We can see that $\hat{D}$ is unitary operator. Moreover, the transformation property of $\hat{D}$ for $\hat{x}$ and $\hat{p}$ can be obtained.

$$\begin{aligned}
\hat{D}\hat{x}\hat{D}^{\dagger}&=\iint_{-\infty}^{\infty}dxdx'|x-x_0\rangle\langle x|\hat{x}|x'\rangle\langle x'-x_0|=\int_{-\infty}^{\infty}xdx|x-x_0\rangle\langle x-x_0|\\
&=\pi^{-1/2}\int_{-\infty}^{\infty}dx:x\exp[-(x-x_0)^2+\sqrt{2}(x-x_0)(\hat{a}^{\dagger}+\hat{a})-(\hat{a}^{\dagger}+\hat{a})^2/2]:\\
&=\hat{x}+x_0,\\
\hat{D}\hat{p}\hat{D}^{\dagger}&=\iint_{-\infty}^{\infty}dxdx'|x-x_0\rangle\langle x|\hat{p}|x'\rangle\langle x'-x_0|\\
&=\iint_{-\infty}^{\infty}dxdx'\left[\mathrm{i}\frac{\partial}{\partial x'}|x-x_0\rangle\delta(x-x')\right]\langle x'-x_0|\\
&=\mathrm{i}\int_{-\infty}^{\infty}dx\left[\frac{\partial}{\partial x}|x-x_0\rangle\right]\langle x-x_0|=\mathrm{i}\int_{-\infty}^{\infty}dx[-(x-x_0)+\sqrt{2}\hat{a}^{\dagger}]|x-x_0\rangle\langle x-x_0|\\
&=\pi^{-1/2}\mathrm{i}\int_{-\infty}^{\infty}dx:[-(x-x_0)+\sqrt{2}\hat{a}^{\dagger}]\exp[-(x-x_0)^2+\sqrt{2}(x-x_0)(\hat{a}^{\dagger}+\hat{a})\\
&\quad-(\hat{a}^{\dagger}+\hat{a})^2/2]:\\
&=\mathrm{i}\left(-\frac{\hat{a}^{\dagger}+\hat{a}}{\sqrt{2}}+\sqrt{2}\hat{a}^{\dagger}\right)=\hat{p}.
\end{aligned}$$

We can see that, $\hat{D}$ is quantum unitary operator corresponding to classical coordinate translation.

In the case of multimode, coordinate displacement operator is

$$\hat{D}=\int_{-\infty}^{\infty}dx_1dx_2\cdots dx_n|x_1-x_{10},x_2-x_{20},\ldots,x_n-x_{n0}\rangle\langle x_1,x_2,\ldots,x_n|. \tag{6}$$

It is easily proved that $\hat{D}\hat{D}^{\dagger}=\hat{D}^{\dagger}\hat{D}=1$, and $\hat{D}\hat{x}_i\hat{D}^{\dagger}=\hat{x}_i+x_{i0}$, $\hat{D}\hat{p}_i\hat{D}^{\dagger}=\hat{p}_i$.

2.3 Squeezing Operator

Based on the classical squeezing transformation $x_i\to kx_i$, we construct the following asymmetric integration form projection operator

$$\hat{S}=\sqrt{k_1\cdots k_n}\iint_{-\infty}^{\infty}dx_1\cdots dx_n|k_1x_1,\ldots,k_nx_n\rangle\langle x_1,\ldots,x_n|. \tag{7}$$

From (7) we obtain

$$\begin{aligned}\hat{S}\hat{S}^{\dagger} &= k_1\cdots k_n \iint_{-\infty}^{\infty} dx_1\cdots dx_n dx_1'\cdots dx_n' |k_1x_1,\ldots,k_nx_n\rangle\langle x_1,\ldots,x_n| \\ &\quad\times |x_1',\ldots,x_n'\rangle\langle k_1x_1',\ldots,k_nx_n'| \\ &= k_1\cdots k_n \iint_{-\infty}^{\infty} dx_1\cdots dx_n |k_1x_1,\ldots,k_nx_n\rangle\langle k_1x_1,\ldots,k_nx_n| = 1 = \hat{S}^{\dagger}\hat{S}.\end{aligned}$$

Thus we know that $\hat{S}$ is a unitary operator.

Moreover, the transformation property of $\hat{S}$ for $\hat{x}_i$ and $\hat{p}_i$ can be obtained. From (7) we have

$$\begin{aligned}\hat{S}\hat{x}_i\hat{S}^{\dagger} &= k_1\cdots k_n \iint_{-\infty}^{\infty} dx_1\cdots dx_n dx_1'\cdots dx_n' |k_1x_1,\ldots,k_nx_n\rangle\langle x_1,\ldots,x_n|\hat{x}_i|x_1',\ldots,x_n'\rangle \\ &\quad\times \langle k_1x_1',\ldots,k_nx_n'| \\ &= k_1\cdots k_n \int_{-\infty}^{\infty} x_i dx_1\cdots dx_n |k_1x_1,\ldots,k_nx_n\rangle\langle k_1x_1,\ldots,k_nx_n| \\ &= k_i \int_{-\infty}^{\infty} x_i dx_i |k_ix_i\rangle\langle k_ix_i| \\ &= \pi^{-1/2}k_i \int_{-\infty}^{\infty} dx_i : x_i \exp[-k_i^2x_i^2 + \sqrt{2}k_ix_i(\hat{a}^{\dagger}+\hat{a}) - (\hat{a}^{\dagger}+\hat{a})^2/2] := \frac{\hat{x}_i}{k_i}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\hat{S}\hat{p}_i\hat{S}^{\dagger} &= k_1\cdots k_n \iint_{-\infty}^{\infty} dx_1\cdots dx_n dx_1'\cdots dx_n' |k_1x_1,\ldots,k_nx_n\rangle\langle x_1,\ldots,x_n|\hat{p}_i|x_1',\ldots,x_n'\rangle \\ &\quad\times \langle k_1x_1',\ldots,k_nx_n'| \\ &= k_1\cdots k_n \iint_{-\infty}^{\infty} dx_1\cdots dx_n dx_1'\cdots dx_n' \\ &\quad\times \left[\mathrm{i}\frac{\partial}{\partial x_i'}|k_1x_1,\ldots,k_nx_n\rangle\delta(x_1-x_1')\cdots\delta(x_n-x_n')\right]\langle k_1x_1',\ldots,k_nx_n'| \\ &= k_1\cdots k_n \iint_{-\infty}^{\infty} dx_1\cdots dx_n \left[\mathrm{i}\frac{\partial}{\partial x_i}|k_1x_1,\ldots,k_nx_n\rangle\right]\langle k_1x_1,\ldots,k_nx_n| \\ &= \mathrm{i}k_i \int_{-\infty}^{\infty} dx_i \cdot (-k_i^2x_i + \sqrt{2}k_ia_i^{\dagger})|k_ix_i\rangle\langle k_ix_i| \\ &= \mathrm{i}\pi^{-1/2}k_i \int_{-\infty}^{\infty} dx_i : (-k_i^2x_i + \sqrt{2}k_ia_i^{\dagger})\exp[-k_i^2x_i^2 + \sqrt{2}k_ix_i(\hat{a}^{\dagger}+\hat{a}) \\ &\quad - (\hat{a}^{\dagger}+\hat{a})^2/2] : \\ &= \mathrm{i}(-k_i\hat{x}_i + \sqrt{2}k_ia_i^{\dagger}) = k_i\hat{p}_i.\end{aligned}$$

We can see that $\hat{S}$ is indeed a n-module operator simultaneously squeezing $\hat{x}_i$ and $\hat{p}_i$.

2.4 Unitary Operator Corresponding to the Classical Canonical Transformation

Based on classical canonical transformation $(q_1, q_2) \to (Aq_1 + Bq_2, Cq_1 + Dq_2)$, we construct the following asymmetric integration

$$\hat{U} = \iint_{-\infty}^{\infty} dq_1 dq_2 \left| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right|, \quad (AD - BC = 1). \tag{8}$$

From (8), we obtain

$$\begin{aligned}
\hat{U}\hat{U}^\dagger &= \int_{-\infty}^{\infty} dq_1 dq_2 dq_1' dq_2' \left| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right| \left| \begin{pmatrix} q_1' \\ q_2' \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} q_1' \\ q_2' \end{pmatrix} \right| \\
&= \int_{-\infty}^{\infty} dq_1 dq_2 \left| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right| \\
&= \pi^{-1} \int_{-\infty}^{\infty} dq_1 dq_2 \exp\{-(Aq_1 + Bq_2)^2 - (Cq_1 + Dq_2)^2 + \sqrt{2}(Aq_1 + Bq_2)\hat{a}_1^\dagger \\
&\quad + \sqrt{2}(Cq_1 + Dq_2)\hat{a}_2^\dagger - (\hat{a}_1^{\dagger 2} + \hat{a}_2^{\dagger 2})/2\} |0, 0\rangle\langle 0, 0| \exp[\sqrt{2}(Aq_1 + Bq_2)\hat{a}_1 \\
&\quad + \sqrt{2}(Cq_1 + Dq_2)\hat{a}_2 - (\hat{a}_1^2 + \hat{a}_2^2)/2] \\
&= \pi^{-1} \int_{-\infty}^{\infty} dq_1 dq_2 : \exp[-(Aq_1 + Bq_2)^2 - (Cq_1 + Dq_2)^2 \\
&\quad + \sqrt{2}(Aq_1 + Bq_2)(\hat{a}_1^\dagger + \hat{a}_1) + \sqrt{2}(Cq_1 + Dq_2)(\hat{a}_2^\dagger + \hat{a}_2) - (\hat{a}_1 + \hat{a}_1^\dagger)^2/2 \\
&\quad - (\hat{a}_2 + \hat{a}_2^\dagger)^2/2] := 1 = \hat{U}^\dagger \hat{U}.
\end{aligned}$$

Thus we know that $\hat{U}$ is a unitary operator. The transformation property of $\hat{U}$ for $\hat{q}_1$ and $\hat{q}_2$ can be obtained. From (8) we have

$$\begin{aligned}
\hat{U}\hat{q}_1\hat{U}^\dagger &= \int_{-\infty}^{\infty} dq_1 dq_2 dq_1' dq_2' \left| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right| \hat{q}_1 \left| \begin{pmatrix} q_1' \\ q_2' \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} q_1' \\ q_2' \end{pmatrix} \right| \\
&= \int_{-\infty}^{\infty} dq_1 dq_2 \cdot q_1 \left| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right| \\
&= \pi^{-1} \int_{-\infty}^{\infty} dq_1 dq_2 : q_1 \exp[-(Aq_1 + Bq_2)^2 - (Cq_1 + Dq_2)^2 \\
&\quad + \sqrt{2}(Aq_1 + Bq_2)(\hat{a}_1^\dagger + \hat{a}_1) + \sqrt{2}(Cq_1 + Dq_2)(\hat{a}_2^\dagger + \hat{a}_2) - (\hat{a}_1 + \hat{a}_1^\dagger)^2/2 \\
&\quad - (\hat{a}_2 + \hat{a}_2^\dagger)^2/2] := D\hat{q}_1 - B\hat{q}_2.
\end{aligned}$$

Similarly, we have $\hat{U}\hat{q}_2\hat{U}^\dagger = A\hat{q}_2 - C\hat{q}_1$.

Moreover, the transformation property of $\hat{U}$ for $\hat{p}_1$ and $\hat{p}_2$ can be obtained. From (8) we have

$$\begin{aligned}
\hat{U}\hat{p}_1\hat{U}^\dagger &= \int_{-\infty}^{\infty} dq_1 dq_2 dq_1' dq_2' \left| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right| \hat{p}_1 \left| \begin{pmatrix} q_1' \\ q_2' \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} q_1' \\ q_2' \end{pmatrix} \right| \\
&= \int_{-\infty}^{\infty} dq_1 dq_2 dq_1' dq_2' \left[\mathrm{i} \frac{\partial}{\partial q_1'} \left| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\rangle \delta(q_1 - q_1') \delta(q_2 - q_2') \right]
\end{aligned}$$

$$\times\left\langle\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}q_1'\\ q_2'\end{pmatrix}\right|$$

$$=\int_{-\infty}^{\infty}dq_1dq_2\left[\mathrm{i}\frac{\partial}{\partial q_1}\left|\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}q_1\\ q_2\end{pmatrix}\right\rangle\right]\left\langle\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}q_1\\ q_2\end{pmatrix}\right|$$

$$=\mathrm{i}\int_{-\infty}^{\infty}dq_1dq_2[-(A^2+C^2)q_1-(AB+CD)q_2+\sqrt{2}A\hat{a}_1^{\dagger}+\sqrt{2}C\hat{a}_2^{\dagger}]$$

$$\times\left|\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}q_1\\ q_2\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}q_1\\ q_2\end{pmatrix}\right|$$

$$=\mathrm{i}\pi^{-1}\int_{-\infty}^{\infty}dq_1dq_2:[-(A^2+C^2)q_1-(AB+CD)q_2+\sqrt{2}A\hat{a}_1^{\dagger}+\sqrt{2}C\hat{a}_2^{\dagger}]$$

$$\times\exp[-(Aq_1+Bq_2)^2-(Cq_1+Dq_2)^2+\sqrt{2}(Aq_1+Bq_2)(\hat{a}_1^{\dagger}+\hat{a}_1)$$

$$+\sqrt{2}(Cq_1+Dq_2)(\hat{a}_2^{\dagger}+\hat{a}_2)-(\hat{a}_1+\hat{a}_1^{\dagger})^2/2-(\hat{a}_2+\hat{a}_2^{\dagger})^2/2]:$$

$$=\mathrm{i}(-A\hat{q}_1-C\hat{q}_2+\sqrt{2}A\hat{a}_1^{\dagger}+\sqrt{2}C\hat{a}_2^{\dagger})=A\hat{p}_1+C\hat{p}_2.$$

Similarly, we have $\hat{U}\hat{p}_2\hat{U}^{\dagger}=B\hat{p}_1+D\hat{p}_2$.

We can see that $\hat{U}$ is a quantum unitary operator corresponding to classical canonical transformation.

We can also construct a quantum unitary operator by use the following asymmetric integration of momentum eigenvector in Fock representation

$$\hat{U}_p=\iint_{-\infty}^{\infty}dp_1dp_2\left|\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}p_1\\ p_2\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}p_1\\ p_2\end{pmatrix}\right|\quad(AD-BC=1).\tag{9}$$

From (9), we obtain $\hat{U}_p\hat{U}_p^{\dagger}=\hat{U}_p^{\dagger}\hat{U}_p=1$. The transformation property of $\hat{U}_p$ for $\hat{q}_1$ and $\hat{q}_2$ and $\hat{p}_1$, $\hat{p}_2$ can be obtained. From (9) we have

$$\hat{U}_p\hat{q}_1\hat{U}_p^{\dagger}=\int_{-\infty}^{\infty}dp_1dp_2dp_1'dp_2'\left|\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}p_1\\ p_2\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}p_1\\ p_2\end{pmatrix}\right|\hat{q}_1\left|\begin{pmatrix}p_1'\\ p_2'\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}p_1'\\ p_2'\end{pmatrix}\right|$$

$$=\int_{-\infty}^{\infty}dp_1dp_2dp_1'dp_2'\left[-\mathrm{i}\frac{\partial}{\partial p_1'}\left|\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}p_1\\ p_2\end{pmatrix}\right\rangle\delta(p_1-p_1')\delta(p_2-p_2')\right]$$

$$\times\left\langle\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}p_1'\\ p_2'\end{pmatrix}\right|$$

$$=\int_{-\infty}^{\infty}dp_1dp_2\left[-\mathrm{i}\frac{\partial}{\partial p_1}\left|\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}p_1\\ p_2\end{pmatrix}\right\rangle\right]\left\langle\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}p_1\\ p_2\end{pmatrix}\right|$$

$$=-\mathrm{i}\int_{-\infty}^{\infty}dp_1dp_2[-A(Ap_1+Bp_2)-C(Cp_1+Dp_2)$$

$$+\mathrm{i}\sqrt{2}A\hat{a}_1^{\dagger}+\mathrm{i}\sqrt{2}C\hat{a}_2^{\dagger}]\left|\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}p_1\\ p_2\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}p_1\\ p_2\end{pmatrix}\right|$$

$$=-\mathrm{i}\pi^{-1}\int_{-\infty}^{\infty}dp_1dp_2:[-(A^2+C^2)p_1-(AB+CD)p_2+\mathrm{i}\sqrt{2}(A\hat{a}_1^{\dagger}+C\hat{a}_2^{\dagger})]$$

$$\times \exp\left[-(Ap_1+Bp_2)^2-(Cp_1+Dp_2)^2-\mathrm{i}\sqrt{2}(Ap_1+Bp_2)(\hat{a}_1-\hat{a}_1^\dagger)\right.$$
$$\left.-\mathrm{i}\sqrt{2}(Cp_1+Dp_2)(\hat{a}_2-\hat{a}_2^\dagger)+\frac{(\hat{a}_1-\hat{a}_1^\dagger)^2}{2}+\frac{(\hat{a}_2-\hat{a}_2^\dagger)^2}{2}\right]:$$
$$=-\mathrm{i}(-A\hat{p}_1-C\hat{p}_2+\mathrm{i}\sqrt{2}A\hat{a}_1^\dagger+\mathrm{i}\sqrt{2}C\hat{a}_2^\dagger)=A\hat{q}_1+C\hat{q}_2.$$

Similarly, we have $\hat{U}_p\hat{q}_2\hat{U}_p^\dagger=B\hat{q}_1+D\hat{q}_2$.

Moreover, the transformation property of $\hat{U}_p$ for $\hat{p}_1$ and $\hat{p}_2$ can be obtained. From (8) we have

$$\hat{U}_p\hat{p}_1\hat{U}_p^\dagger=\int_{-\infty}^{\infty}dp_1dp_2dp_1'dp_2'\left|\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}p_1\\ p_2\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}p_1\\ p_2\end{pmatrix}\right|\hat{p}_1\left|\begin{pmatrix}p_1'\\ p_2'\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}p_1'\\ p_2'\end{pmatrix}\right|$$
$$=\int_{-\infty}^{\infty}dp_1dp_2p_1\left|\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}p_1\\ p_2\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}p_1\\ p_2\end{pmatrix}\right|$$
$$=\pi^{-1}\int_{-\infty}^{\infty}dp_1dp_2:p_1\exp\left[-(Ap_1+Bp_2)^2-(Cp_1+Dp_2)^2\right.$$
$$-\mathrm{i}\sqrt{2}(Ap_1+Bp_2)(\hat{a}_1-\hat{a}_1^\dagger)-\mathrm{i}\sqrt{2}(Cp_1+Dp_2)(\hat{a}_2-\hat{a}_2^\dagger)$$
$$\left.+\frac{(\hat{a}_1-\hat{a}_1^\dagger)^2}{2}+\frac{(\hat{a}_2-\hat{a}_2^\dagger)^2}{2}\right]:=D\hat{p}_1-B\hat{p}_2.$$

Similarly, we have $\hat{U}_p\hat{p}_2\hat{U}_p^\dagger=A\hat{p}_2-C\hat{p}_1$.

We can see that $\hat{U}_p$ is also a quantum unitary operator corresponding to classical canonical transformation.

2.5 Unitary Operator Corresponding to Classical Symplectic Transformation

Based on classical symplectic transformation $\binom{q_1}{p_2}\to\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)\binom{q_1}{p_2}$, we construct the following asymmetric integration

$$\hat{U}=\iint_{-\infty}^{\infty}dq_1dp_2\left|\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}q_1\\ p_2\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}q_1\\ p_2\end{pmatrix}\right|\quad(AD-BC=1).\tag{10}$$

From (10), we obtain

$$\hat{U}\hat{U}^\dagger=\int_{-\infty}^{\infty}dq_1dp_2dq_1'dp_2'\left|\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}q_1\\ p_2\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}q_1\\ p_2\end{pmatrix}\right|\left|\begin{pmatrix}q_1'\\ p_2'\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}q_1'\\ p_2'\end{pmatrix}\right|$$
$$=\int_{-\infty}^{\infty}dq_1dp_2\left|\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}q_1\\ p_2\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}q_1\\ p_2\end{pmatrix}\right|$$
$$=\pi^{-1}\int_{-\infty}^{\infty}dq_1dp_2\exp\left[-(Aq_1+Bp_2)^2-(Cq_1+Dp_2)^2+\sqrt{2}(Aq_1+Bp_2)\hat{a}_1^\dagger\right.$$
$$\left.+\mathrm{i}\sqrt{2}(Cq_1+Dp_2)\hat{a}_2^\dagger-\frac{\hat{a}_1^{\dagger 2}}{2}+\frac{\hat{a}_2^{\dagger 2}}{2}\right]|0,0\rangle\langle 0,0|\exp\left[\sqrt{2}(Aq_1+Bp_2)\hat{a}_1\right.$$
$$\left.-\mathrm{i}\sqrt{2}(Cq_1+Dp_2)\hat{a}_2-\frac{\hat{a}_1^2}{2}+\frac{\hat{a}_2^2}{2}\right]$$

$$
\begin{aligned}
&= \pi^{-1}\int_{-\infty}^{\infty} dq_1 dp_2 : \exp\Big[-(Aq_1+Bp_2)^2-(Cq_1+Dp_2)^2 \\
&\quad + \sqrt{2}(Aq_1+Bp_2)(\hat{a}_1^\dagger+\hat{a}_1) + \mathrm{i}\sqrt{2}(Cq_1+Dp_2)(\hat{a}_2^\dagger-\hat{a}_2) \\
&\quad - \frac{(\hat{a}_1+\hat{a}_1^\dagger)^2}{2} + \frac{(\hat{a}_2-\hat{a}_2^\dagger)^2}{2}\Big] := 1 = \hat{U}^\dagger\hat{U}.
\end{aligned}
$$

Thus we know that $\hat{U}$ is a unitary operator. The transformation property of $\hat{U}$ for $\hat{q}_1$ and $\hat{q}_2$ can be obtained. From (10) we have

$$
\begin{aligned}
\hat{U}\hat{q}_1\hat{U}^\dagger &= \int_{-\infty}^{\infty} dq_1 dp_2 dq_1' dp_2' \left|\begin{pmatrix} A & B \\ C & D\end{pmatrix}\begin{pmatrix} q_1 \\ p_2\end{pmatrix}\right\rangle\left\langle\begin{pmatrix} q_1 \\ p_2\end{pmatrix}\right|\hat{q}_1\left|\begin{pmatrix} q_1' \\ p_2'\end{pmatrix}\right\rangle\left\langle\begin{pmatrix} A & B \\ C & D\end{pmatrix}\begin{pmatrix} q_1' \\ p_2'\end{pmatrix}\right| \\
&= \int_{-\infty}^{\infty} dq_1 dp_2 \cdot q_1 \left|\begin{pmatrix} A & B \\ C & D\end{pmatrix}\begin{pmatrix} q_1 \\ p_2\end{pmatrix}\right\rangle\left\langle\begin{pmatrix} A & B \\ C & D\end{pmatrix}\begin{pmatrix} q_1 \\ p_2\end{pmatrix}\right| \\
&= \pi^{-1}\int_{-\infty}^{\infty} dq_1 dp_2 \cdot q_1 : \exp\Big[-(Aq_1+Bp_2)^2-(Cq_1+Dp_2)^2 \\
&\quad + \sqrt{2}(Aq_1+Bp_2)(\hat{a}_1^\dagger+\hat{a}_1) + \mathrm{i}\sqrt{2}(Cq_1+Dp_2)(\hat{a}_2^\dagger-\hat{a}_2) \\
&\quad - \frac{(\hat{a}_1+\hat{a}_1^\dagger)^2}{2} + \frac{(\hat{a}_2-\hat{a}_2^\dagger)^2}{2}\Big] := D\hat{q}_1 - B\hat{p}_2,
\end{aligned}
$$

$$
\begin{aligned}
\hat{U}\hat{q}_2\hat{U}^\dagger &= \int_{-\infty}^{\infty} dq_1 dp_2 dq_1' dp_2' \left|\begin{pmatrix} A & B \\ C & D\end{pmatrix}\begin{pmatrix} q_1 \\ p_2\end{pmatrix}\right\rangle\left\langle\begin{pmatrix} q_1 \\ p_2\end{pmatrix}\right|\hat{q}_2\left|\begin{pmatrix} q_1' \\ p_2'\end{pmatrix}\right\rangle\left\langle\begin{pmatrix} A & B \\ C & D\end{pmatrix}\begin{pmatrix} q_1' \\ p_2'\end{pmatrix}\right| \\
&= \int_{-\infty}^{\infty} dq_1 dp_2 dq_1' dp_2' \left[-\mathrm{i}\frac{\partial}{\partial p_2'}\left|\begin{pmatrix} A & B \\ C & D\end{pmatrix}\begin{pmatrix} q_1 \\ p_2\end{pmatrix}\right\rangle \delta(q_1-q_1')\delta(p_2-p_2')\right] \\
&\quad \times \left\langle\begin{pmatrix} A & B \\ C & D\end{pmatrix}\begin{pmatrix} q_1' \\ p_2'\end{pmatrix}\right| \\
&= \int_{-\infty}^{\infty} dq_1 dp_2 \cdot \left[-\mathrm{i}\frac{\partial}{\partial p_2}\left|\begin{pmatrix} A & B \\ C & D\end{pmatrix}\begin{pmatrix} q_1 \\ p_2\end{pmatrix}\right\rangle\right]\left\langle\begin{pmatrix} A & B \\ C & D\end{pmatrix}\begin{pmatrix} q_1 \\ p_2\end{pmatrix}\right| \\
&= -\mathrm{i}\int_{-\infty}^{\infty} dq_1 dp_2[-B(Aq_1+Bp_2) - D(Cq_1+Dp_2) + \sqrt{2}B\hat{a}_1^\dagger + \sqrt{2}iD\hat{a}_2^\dagger] \\
&\quad \times \left|\begin{pmatrix} A & B \\ C & D\end{pmatrix}\begin{pmatrix} q_1 \\ p_2\end{pmatrix}\right\rangle\left\langle\begin{pmatrix} A & B \\ C & D\end{pmatrix}\begin{pmatrix} q_1 \\ p_2\end{pmatrix}\right| \\
&= -\mathrm{i}\pi^{-1}\int_{-\infty}^{\infty} dq_1 dp_2 : [-B(Aq_1+Bp_2) - D(Cq_1+Dp_2) + \sqrt{2}B\hat{a}_1^\dagger + \sqrt{2}\mathrm{i}D\hat{a}_2^\dagger] \\
&\quad \times \exp\Big[-(Aq_1+Bp_2)^2-(Cq_1+Dp_2)^2 + \sqrt{2}(Aq_1+Bp_2)(\hat{a}_1^\dagger+\hat{a}_1) \\
&\quad + \mathrm{i}\sqrt{2}(Cq_1+Dp_2)(\hat{a}_2^\dagger-\hat{a}_2) - \frac{(\hat{a}_1+\hat{a}_1^\dagger)^2}{2} + \frac{(\hat{a}_2-\hat{a}_2^\dagger)^2}{2}\Big] : \\
&= -\mathrm{i}[-B\hat{q}_1 - D\hat{p}_2 + \sqrt{2}B\hat{a}_1^\dagger + \sqrt{2}\mathrm{i}D\hat{a}_2^\dagger] = D\hat{q}_2 - B\hat{p}_1.
\end{aligned}
$$

Similarly, we have $\hat{U}\hat{p}_1\hat{U}^\dagger = A\hat{p}_1 - C\hat{p}_2$, $\hat{U}\hat{p}_2\hat{U}^\dagger = A\hat{p}_2 - C\hat{q}_1$.

We can see that $\hat{U}$ is a quantum unitary operator corresponding to classical symplectic transformation.

3 Energy Eigenvalue of the Two-Mode Momentum Coupling Harmonic Osillator

Consider a two-mode momentum coupling harmonic oscillator system as an example, whose Hamiltonian reads

$$\hat{H} = \frac{1}{2m}(\hat{p}_1^2 + \hat{p}_2^2) + \frac{1}{2}m\omega^2(\hat{x}_1^2 + \hat{x}_2^2) + \gamma\hat{p}_1\hat{p}_2. \tag{11}$$

In order to exactly solve the energy eigenvalue of the system, it is necessary to find a way to diagonalize the Hamiltonian so as to eliminate coupling. The unitary transformation property of $\hat{U}_p$ in (9) is employed to diagonalize $\hat{H}$

$$\begin{aligned}\hat{U}_p\hat{H}\hat{U}_p^\dagger &= \frac{1}{2m}[(D\hat{p}_1 - B\hat{p}_2)^2 + (A\hat{p}_2 - C\hat{p}_1)^2] + \frac{1}{2}m\omega^2[(A\hat{x}_1 + C\hat{x}_2)^2 + (B\hat{x}_1 + D\hat{x}_2)^2] \\ &\quad + \gamma(D\hat{p}_1 - B\hat{p}_2)(A\hat{p}_2 - C\hat{p}_1) \\ &= \left[\frac{1}{2m}(C^2 + D^2) - \gamma CD\right]\hat{p}_1^2 + \left[\frac{1}{2m}(A^2 + B^2) - \gamma AB\right]\hat{p}_2^2 \\ &\quad + \frac{1}{2}m\omega^2(A^2 + B^2)\hat{x}_1^2 + \frac{1}{2}m\omega^2(C^2 + D^2)\hat{x}_2^2 \\ &\quad + \left[-\frac{AC + BD}{m} + \gamma(AD + BC)\right]\hat{p}_1\hat{p}_2 + m\omega^2(AC + BD)\hat{x}_1\hat{x}_2. \end{aligned} \tag{12}$$

Defining zeroth coefficient of $\hat{p}_1\hat{p}_2$ and $\hat{x}_1\hat{x}_2$, we choose $A = 1$, $B = 1$, $C = -1/2$, $D = 1/2$. The unitary operator $\hat{U}_p$ can be written as

$$\hat{U}_p = \iint_{-\infty}^{\infty} dp_1 dp_2 \left|\begin{pmatrix} 1 & 1 \\ -1/2 & 1/2 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}\right\rangle\left\langle\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}\right|.$$

This is can be find that $\hat{U}_p$ includes rotation transformation and squeezing transformation. From (12), we obtain

$$\begin{aligned}\hat{U}_p\hat{H}\hat{U}_p^\dagger &= \frac{1 + m\gamma}{4m}\hat{p}_1^2 + \frac{1 - m\gamma}{m}\hat{p}_2^2 + m\omega^2\hat{x}_1^2 + \frac{1}{4}m\omega^2\hat{x}_2^2 \\ &= \frac{\hat{p}_1^2}{2M_1} + \frac{\hat{p}_2^2}{2M_2} + \frac{1}{2}M_1\Omega_1^2 + \frac{1}{2}M_2\Omega_2^2, \end{aligned} \tag{13}$$

where

$$M_1 = \frac{2m}{1 + m\gamma}, \qquad M_2 = \frac{m}{2(1 - m\gamma)}, \qquad \Omega_1 = \omega(1 + m\gamma)^{1/2}, \qquad \Omega_2 = \omega(1 - m\gamma)^{1/2}.$$

Because the transformation to $\hat{H}$ is unitary transform, the energy level distribution of system did not change. Comparising with one-dimension Harmonic oscillator, we can obtain the energy eigenvalue

$$E_{\mathrm{n}_1,\mathrm{n}_2} = \left(n_1 + \frac{1}{2}\right)\hbar\Omega_1 + \left(n_2 + \frac{1}{2}\right)\hbar\Omega_2 \quad n_1, n_2 = 0, 1, 2, \ldots.$$

When $\gamma = 0$, then $\Omega_1 = \Omega_2 = \omega$,

$$E_{\mathrm{n}_1,\mathrm{n}_2} = \left(n_1 + \frac{1}{2}\right)\hbar\omega + \left(n_2 + \frac{1}{2}\right)\hbar\omega \quad n_1, n_2 = 0, 1, 2, \ldots.$$

It is the case of no coupling.

4 Conclusions

Based on known coordinate eigenvector and momentum eigenvector in Fock representation, Some unitary operators are derived using quantum state by depending on the technique of integration within an ordered product of operators, for example parity operator, displacement operator, squeezed operator, etc. These unitary operators are corresponding to some classical transformation, and its characteristics are analyzed. Thus the transient from some classical transformation to quantum unitary transform is realized. Their unitary transformations play an essential role in some transformations by using IWOP technique. As applications, the dynamic problems of the double momentum coupling harmonic oscillators are solved exactly.

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